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Procedia Engineering 23 (2011) 387 – 391
Procedia Engineering
www.elsevier.com/locate/procedia

Power Electronics and Engineering Application

A Simple Method for Solving Multiperiod Mean-Variance Asset-Liability Management Problem

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Abstract

This paper introduces the Lagrange duality method for solving the multiperiod mean-variance (M-V) asset-liability management (ALM) problem. First, Using the Lagrange multiplier technique, the original problem is turned into a multi-period unconstrained Optimal Control Problem (OCP) that is separable in the sense of dynamic programming. Then the dynamic programming approach is applied to solve the OCP. Finally, closed form expressions of the efficient investment strategy and the M-V efficient frontier are obtained.

Selection and/or peer-review under responsibility of [name organizer]

Keywords: asset-liability management; multi-period mean-variance; Lagrange duality; dynamic programming

1. Introduction

Since the pioneer work of Markowitz[1], the mean–variance (M–V) model has became the foundation of modern finance theory and inspired hundreds of extensions and applications. Among them, [2] and [3] extend the model to cases of multi-period and continuous-time, respectively, by using an embedding technique that overcame the difficulty of non-separability in variance, and derived the analytical optimal solutions. After that, many scholars adopt the dynamic M–V model to study other portfolio selection or financial problems under some reality conditions (see [4] for example).

On the other hand, it is well known that asset-liability management (ALM) problem is of both theoretical interest and practical importance. For example, ALM has extensive applications in banks,

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doi:10.1016/j.proeng.2011.11.2518

pension funds and insurance companies. In recent years, using the M–V criteria, [5] studied a multiperiod ALM problem. [6] and [7] extended the work of [5] to the cases of uncertain exit time and stochastic market environment, respectively. On the other hand, [8]-[9] investigated ALM problems under M–V criterion in continuous-time setting.

To the best of our knowledge, Most the existing literatures about multiperiod M–V model with liability apply the embedding techniques introduced in [2] to solve the model. Though embedding techniques overcome the difficulty of non-separability in variance, but it's quite complicated in procedure settings and calculation. What is more, except for wealth, ALM problems need to consider another relevant state variable, liability, the inclusion of further state variable drastically enhances the computational complexity in obtaining closed form solutions (refer to [5]-[7]). For these reasons, this paper tries to introduce a new simple method, that named as Lagrange duality method, for solving multiperiod mean–variance ALM problems. Compared with the embedding techniques, Lagrange duality methods are simpler in procedure settings with less computational complexity. As an application and a demonstration of Lagrange duality method, This paper adopt it to solve the multiperiod mean–variance ALM problems of [5].

The paper is structured as follows. Section two sets up the mean-variance ALM problem as a multi-period Optimal Control Problem (OCP) with equality constraint. In section three, the original problem is turned into a separable multi-period unconstraint OCP by using Lagrange multiplier technique and the analytical solution is obtained by dynamic programming approach. Finally, the efficient investment strategy and the efficient M-V frontier are obtained in section four.

2. Establishment of mean–variance ALM model

Suppose that there are $n+1$ securities with return vector $e_k=(e_k^0, e_k^1, e_k^2,\cdots,e_k^n)'$ at time period k, $k=0,1,\cdots,T$. Here A' represents the transpose of matrix A. An investor, equipped with initial wealth x_0 and initial liability l_0, enters the market at time 0, and makes investments within T period. He (she) not only need to consider the investment strategy, but also consider the liability management. Following [5], the liability are uncontrollable, and its dynamic process is

$$l_{k+1}=q_k l_k\ , \tag{1}$$

where q_k is a exogenous random variable, it can be understood as the random growth rate of liability. Let x_k and l_k denote the value of wealth and liability he holds at period k, respectively. u_k^i, $i=1,2,\cdots,n$, is the amount invested in the i th security at period k, then the amount invested in the 0 th security is $x_k-\sum_{i=1}^{n}u_k^i$. Therefore, the wealth dynamics can be written as (see [5])

$$x_{k+1}=x_k e_k^0+P_k' u_k, \tag{2}$$

where $P_k=(e_k^1-e_k^0, e_k^2-e_k^0,\cdots,e_k^n-e_k^0)'$. This paper has the assumptions as [5].

$\wp_k$ is the overall information sets till time k. Then the investment strategy u_k is admissible if u_k is adapted to $\wp_k$. The collection of all admissible investment strategy is defined as Θ_k.

The multiperiod mean-variance ALM problems is to find out the optimal admissible strategy to minimize the risk of the final surplus, defined as $S_T=x_T-l_T$, under the condition that expectation is given as d, here the risk is measured by variance, i.e. $Var[S_T]=E[S_T^2]-E^2[S_T]=E[S_T^2]-d^2$.

Therefore, the multiperiod ALM model under the M-V framework can be now formulated as OCP:

$$\min_{u_k\in\Theta_k} Var[S_T]=E[S_T^2]-d^2, \quad s.t. \quad E[S_T]=d,(1)-(2). \tag{3}$$

The solution of this OCP is called **efficient investment strategy**. The collection of all these points $(d, Var[S_T])$ in coordinate plane M-V corresponded to efficient strategies is called as **efficient frontier**.

3. Transformation and solution to the problem

It is well known that the equality constraint $E[S_T]=d$ in OCP (3) can be dealt with by introducing a Lagrange multiplier μ. We can turn to solve the following unconstrained OCP parameterized by μ

$$\min_{u_k\in\Theta_k} E[S_T^2]-d^2+2\mu(E[S_T]-d), \quad s.t. \quad (1)-(2). \tag{4}$$

Since

$$E[S_T^2]-d^2+2\mu(E[S_T]-d)=E[(x_T-l_T)^2+2\mu(x_T-l_T)]-d^2-2\mu d.$$

Therefore, OCP (4) is equivalent to

$$\min_{u_k\in\Theta_k} E[x_T{}^2+l_T{}^2-2x_Tl_T+2\mu x_T-2\mu l_T-d^2-2\mu d], \quad s.t. \quad (1)-(2). \tag{5}$$

In the following, we solve OCP (5) by using dynamic programming approach.

Let $f_k(x_k,l_k)$ denote the optimal value function associated with OC (5) starting from time k with state: wealth x_k and liability l_k. Then, according to the dynamic programming principle, the basic equations of OC (5) are as follows:

$$\begin{cases} f_k(x_k,l_k)=\min\limits_{u_k\in\Theta_k} E\left[f_{k+1}(x_ke_k^0+P_k'u_k,q_kl_k)\right], \\ f_T(x_T,l_T)=x_T{}^2+l_T{}^2-2x_Tl_T+2\mu x_T-2\mu l_T-d^2-2\mu d. \end{cases} \tag{6}$$

As a result, $H(x_0,l_0,\lambda):=f_0(x_0,l_0)$ is the optimal value of OCP (5).

For simplicity, let $x=x_k$, $l=l_k$. We guess and subsequently verify that the expression of $f_k(x)$ has the form as follows

$$f_k(x,l)=\frac{1}{2}w_kx^2+\lambda_kxl+\gamma_kl^2+h_kx+g_kl+\alpha_k, \tag{7}$$

where $w_k>0$, $\lambda_k,\gamma_k,h_k,g_k,\alpha_k$ are series to be determined.

Substituting (7) into the first equation of basic equation (6) gives

$$\begin{aligned} &\frac{1}{2}w_kx^2+\lambda_kxl+\gamma_kl^2+h_kx+g_kl+\alpha_k=\min_{u_k} E\left[f_{k+1}(xe_k^0+P_k'u_k,q_kl)\right] \\ &=\min_{u_k} E\left\{\frac{1}{2}w_{k+1}(xe_k^0+P_k'u_k)^2+\lambda_{k+1}(xe_k^0+P_k'u_k)q_kl+\gamma_{k+1}q_k^2l^2+h_{k+1}(xe_k^0+P_k'u_k)+g_{k+1}q_kl+\alpha_{k+1}\right\} \\ &=\frac{1}{2}w_{k+1}E[(e_k^0)^2]x^2+\lambda_{k+1}E[e_k^0q_k]xl+\gamma_{k+1}E[q_k^2]l^2+h_{k+1}E[e_k^0]x+g_{k+1}E[q_k]l+\alpha_{k+1} \\ &\qquad +\min_{u_k}\left\{\frac{1}{2}w_{k+1}u_k'E[P_kP_k']u_k+(w_{k+1}E[e_k^0P_k']x+\lambda_{k+1}E[q_kP_k']l+h_{k+1}E[P_k'])u_k\right\}. \end{aligned}$$

The first order condition (since $w_k>0$, then is also sufficient condition) gives

$$u_k=-E^{-1}[P_kP_k']\left(E[e_k^0P_k]x+\frac{\lambda_{k+1}}{w_{k+1}}E[q_kP_k]l+\frac{h_{k+1}}{w_{k+1}}E[P_k]\right), \tag{8}$$

substituted back into the above formula, it follows that

$$\frac{1}{2}w_k x^2+\lambda_k xl+\gamma_k l^2+h_k x+g_k l+\alpha_k$$
$$=\frac{1}{2}w_{k+1}A_k x^2+\lambda_{k+1}G_k xl+\left(\gamma_{k+1}E[q_k^2]-\frac{1}{2}\frac{\lambda_{k+1}^2}{w_{k+1}}B_k\right)l^2+h_{k+1}J_k x+\left(g_{k+1}E[q_k]-\frac{\lambda_{k+1}h_{k+1}}{w_{k+1}}M_k\right)l+\alpha_{k+1}-\frac{1}{2}\frac{h_{k+1}^2}{w_{k+1}}D_k,$$

where

$$\begin{cases} A_k=E[(e_k^0)^2]-E[e_k^0P_k']E^{-1}[P_kP_k']E[e_k^0P_k],\ B_k=E[q_kP_k']E^{-1}[P_kP_k']E[q_kP_k],\\ D_k=E[P_k']E^{-1}[P_kP_k']E[P_k],G_k=E[e_k^0q_k]-[e_k^0P_k']E^{-1}[P_kP_k']E[q_kP_k],\\ J_k=E[e_k^0]-[e_k^0P_k']E^{-1}[P_kP_k']E[P_k],\ M_k=E[q_kP_k']E^{-1}[P_kP_k']E[P_k].\end{cases}\tag{9}$$

Therefore, we obtain the recursion relationship about w_k, $\lambda_k,\gamma_k,h_k,g_k,\alpha_k$ as

$$\begin{cases} w_k=w_{k+1}A_k,\ \lambda_k=\lambda_{k+1}G_k,h_k=h_{k+1}J_k,\\ \gamma_k=\gamma_{k+1}E[q_k^2]-\frac{1}{2}\frac{\lambda_{k+1}^2}{w_{k+1}}B_k,\ g_k=g_{k+1}E[q_k]-\frac{\lambda_{k+1}h_{k+1}}{w_{k+1}}M_k,\alpha_k=\alpha_{k+1}-\frac{1}{2}\frac{h_{k+1}^2}{w_{k+1}}D_k.\end{cases}\tag{10}$$

First, by means of repeatedly iteration and notice that $w_T=2,\ \lambda_T=-2,h_T=2\mu$ from (6), we obtain

$$w_k=2F_k,\ \lambda_k=-2C_k,h_k=2\mu L_k,\quad k=0,1,\cdots,T-1,\tag{11}$$

where

$$F_k=\prod_{i=k}^{T-1}A_i,C_k=\prod_{i=k}^{T-1}G_i,L_k=\prod_{i=k}^{T-1}J_i,\quad k=0,1,\cdots,T.\tag{12}$$

Here, we set that $\prod_{i=T}^{T-1}(\bullet)=1$. It is known from [2] that $A_k=E[(e_k^0)^2]-E[e_k^0P_k']E^{-1}[P_kP_k']E[e_k^0P_k]>0.$ Thereby $w_k>0$, which satisfies the previous assumption.

Substituting (11) into (10), then the recursion formula about γ_k,g_k,α_k can be rewritten as

$$\gamma_k=\gamma_{k+1}E[q_k^2]-C_k^2B_kF_k^{-1},\ g_k=g_{k+1}E[q_k]+2\mu L_kM_kF_k^{-1},\ \alpha_k=\alpha_{k+1}-\mu^2L_k^2D_kF_k^{-1}.\tag{13}$$

After repeatedly iterating and note that $\gamma_T=1,\ g_T=-2\mu,\alpha_T=-d^2-2\mu d$ from (6), we get

$$\gamma_k=\prod_{i=k}^{T-1}E[q_i^2]-\sum_{i=k}^{T-1}C_i^2B_iF_i^{-1}\prod_{j=k}^{i-1}E[q_j^2],\ g_k=2\mu I_k,\ \alpha_k=-d^2-2\mu d-\mu^2N_k,\tag{14}$$

where

$$I_k=-\prod_{i=k}^{T-1}E[q_i]+\sum_{i=k}^{T-1}L_iM_iF_i^{-1}\prod_{j=k}^{i-1}E[q_j],\ N_k=\sum_{i=k}^{T-1}L_i^2D_iF_i^{-1},\quad k=0,1,\cdots,T.\tag{15}$$

Here, we define $\sum_{i=T}^{T-1}(\bullet)=0$. As a result, the solution to basic equation (6) is

$$f_k(x_k,l_k)=F_kx_k^{\,2}-2C_kx_kl_k+\gamma_kl_k^{\,2}+2\mu L_kx_k+2\mu I_kl_k-d^2-2\mu d-\mu^2N_k,\tag{16}$$

and the optimal investment strategy is

$$u_k=-E^{-1}[P_kP_k']\left(E[e_k^0P_k]x-C_{k+1}F_{k+1}^{-1}E[q_kP_k]l+\mu L_{k+1}F_{k+1}^{-1}E[P_k]\right).\tag{17}$$

4. The efficient investment strategy and efficient frontier

From previous analysis, we know that when $k=0$, $f_0(x_0,l_0)$ is the optimal value of OCP (5), i.e.

$$H(x_0,l_0,\mu):=f_0(x_0,l_0)=F_0x_0^{\,2}-2C_0x_0l_0+\gamma_0l_0^{\,2}+2\mu L_0x_0+2\mu I_0l_0-d^2-2\mu d-\mu^2N_0\tag{18}$$

It is well known from the Lagrange duality theorem (see [13]) that the optimal value of OCP (3) , namely minimum variance can be obtained by solving the maximum of $H(x_0, l_0, \mu)$ about μ , i.e.

$$Var[x_T] = \max_{\mu} H(x_0, l_0, \mu) = \max_{\mu} \left\{ -N_0\mu^2 + 2(L_0x_0 + I_0l_0 - d)\mu + F_0{x_0}^2 - 2C_0x_0l_0 + \gamma_0{l_0}^2 - d^2 \right\}. \tag{19}$$

Obviously, $N_0 > 0$, thereby, the maximum to OCP (19) exists, and its first-order condition gives the maximum point as

$$\mu^* = N_0^{-1}\left(L_0x_0 + I_0l_0 - d\right), \tag{20}$$

substituted into (17) and note that $x = x_k, l = l_k$, we obtain the optimal strategy to OCP (3) , namely the efficient investment strategy as

$$u_k = -E^{-1}[P_kP_k']\left(E[e_k^0P_k]x_k - C_{k+1}F_{k+1}^{-1}E[q_kP_k]l_k + N_0^{-1}(L_0x_0 + I_0l_0 - d)L_{k+1}F_{k+1}^{-1}E[P_k]\right). \tag{21}$$

Again substituting (20) into (19), we obtain the optimal value of OCP (3), namely minimum variance as

$$\begin{aligned} Var[x_T] &= N_0^{-1}(L_0x_0 + I_0l_0 - d)^2 + F_0{x_0}^2 - 2C_0x_0l_0 + \gamma_0{l_0}^2 - d^2 \\ &= (1 - N_0)N_0^{-1}\left(d - (1 - N_0)^{-1}(L_0x_0 + I_0l_0)\right)^2 + F_0{x_0}^2 - 2C_0x_0l_0 + \gamma_0{l_0}^2 - (1 - N_0)^{-1}(L_0x_0 + I_0l_0)^2. \end{aligned} \tag{22}$$

So far, we obtain the following results.

Theorem 1: For given expected terminal surplus $E[S_T] = d$ in the multi-period mean-variance ALM model, the efficient investment strategy can be obtained by (21), while, the efficient M-V frontier can be given by (22), here $d \geq (1 - N_0)^{-1}(L_0x_0 + I_0l_0)$.

Acknowledgements

This research is supported by the Humanity and Social Science Foundation of Ministry of Education of China (No. 10YJC790339), Natural Science Foundation of Guangdong Province (No. S2011010005503), and Philosophy and Social Science Foundation of Guangdong Province (No. 09O-19).

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